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Abstract

We give the definition of Lazard and Hall sets in the context of transitive factorizations of free monoids. The equivalence of the two properties is proved. This allows to build new effective bases of free partially commutative Lie algebras. The commutation graphs for which such sets exist are completely characterized and we explicit, in this context, the classical PBW rewriting process.

1 Introduction

The correspondence with Lie algebras is not so clear for monoids as it is in the case of groups (Lie groups and, after the work of Magnus [2] and Lazard

[14] combinatorial groups). In fact, the first connection between the two realms was done by means of the notion of factorization [18].

An ordered family $(M_i)_{i\in I}$ of submonoid is said to be a factorization if the product mapping $\coprod_{i\in I} M_i \to M$ is one to one (see paragraph 2.2). For $|I| = 2$, one gets the notion of a bisection related to the flip-flop [1] Lie-algebra by Viennot [18]. On the other end, when all the M_i have a single generator, one obtains a complete factorisation whose generating series is equal to the Hilbert series of the Free Lie algebra. It has been shown in [6] that this property stills holds in case of partial commutations and the link between the Lyndon basis and Lazard elimination in this context has been elucidated [12]. Here, we consider a more general construction: the transitive Hall sets.

The structure of the paper is the following. In section (2), we give a partially commutative version of Schützenberger's factorization theorem. We introduce in Section (3) the notions of transitive Lazard and transitive Hall sets in the free algebras of trees. Transitive Lazard sets are defined by means of iterations of transitive bisections [9]. Transitive Hall sets classically use a description of the total order of the factorization which, here, must be compatible with the commutations. We prove that the two notions coincide and allow the construction of bases of $L(A,\theta)$ as well as a natural Poincaré-Birkhoff-Witt's rewriting process (PBW).

2 Factorizations of a trace monoid

Let A be an alphabet and $\theta \in A\times A - \{(a,a)|a\in A\}$ be a symmetric relation. We will denote by $\mathbb{M}(A,\theta)$ a trace monoid (or a free partially commutative monoid) over the alphabet A and whose commutations are defined by $ab = ba$ when $(a,b)\in\theta$.

The only way to get the equality

$$\underline{\mathbb{M}(A,\theta)} = \underline{a^*}.\underline{\mathbb{M}(X,\theta_X)} \tag{1}$$

where X a subset of $\mathbb{M}(A,\theta)$ is to impose for each pair of letters $(z_1,z_2)\in \theta\cap(A-a)^2$ the commutation $(z_1,a)\in\theta$ or $(z_2,a)\in\theta$. It is a direct consequence of the transitive factorization theorem (see Section (3.3) [9]). In this paper, we use this property to define transitive Lazard sets for a certain family of trace monoids. Let us first show some properties about conjugacy.

[1] In french "bascule"

2.1 Roots of conjugacy classes

Let $m \in \mathbb{M}(A, \theta)$. It is shown in [8] (see also [5]) that the equation $u = t^p$ ($p \geq 1$) has at most one solution. When it exists, this solution will be denoted by $\sqrt[p]{u}$. In the same paper [8], it is shown that if g, r, and t are tree traces such that $g = r^q = t^p$ with $q, p \in \mathbb{N}$ then it exists a trace g' and $m \in \mathrm{lcm}(p, q)\mathbb{N}$ such that $g = g'^m$. Hence, one defines the **root** $\sqrt{g}$ of a trace g as the smallest trace g' satisfying $g = g'^m$ (the integer m will be called the **exponent** of g, we denote $m = \mathrm{ex}(g)$).

Example 2.1 *Let us consider the commutation graph*

$$(A, \theta) = a - c - b$$

we have $\sqrt{babcac} = bac$ *and* $\mathrm{ex}(babcac) = 2$.

We recall here the definition of conjugacy due to Duboc and Choffrut [3]. Two traces t and t' are said to be conjugate if there exists a trace u such that $tu = ut'$. Conjugacy is an equivalence relation which, in turn, is no more that the restriction to $\mathbb{M}(A, \theta)$ of the conjugacy relation of the group $\mathbb{F}(A, \theta)$. Exponents and roots are invariant under conjugacy in the following sense.

Proposition 2.2 *Let C be a conjugacy class, $f \in C$, $g \in \mathbb{M}(A, \theta)$ and $p \in \mathbb{N}$ be such that $f = g^p$. For each $f' \in C$, it exists $g' \in \mathbb{M}(A, \theta)$ such that $f' = g'^p$. Furthermore g and g' are conjugate.*

As a consequence of Proposition (2.2) root $\sqrt{C}$ of a conjugacy class C is uniquely defined as the class $\sqrt{C} = \{\sqrt{g}\}_{g \in C}$.

2.2 Schützenberger's factorization theorem for traces

Let $(M_i)_{i \in J}$ be a family of monoids. The restricted product $\coprod_{i \in J} M_i$ is the submonoid of $\prod_{i \in J} M_i$ of the families with finite support (all but a finite number of indices are 1_{M_i}). Now, if the $(M_i)_{i \in J}$ are submonoids of a given monoid M and if J is totally ordered, the product $\mathrm{prod} : \mathrm{M} \times \mathrm{M} \to \mathrm{M}$ extends to an arrow $\mathrm{prod} : \coprod_{\mathrm{i} \in \mathrm{J}} \mathrm{M_i} \to \mathrm{M}$ by means of the ordered product.
A factorization of a monoid $\mathbb{M}$ is an ordered family of submonoids $(M_i)_{i \in J}$ such that $\mathrm{prod} : \coprod_{\mathrm{i} \in \mathrm{J}} \mathrm{M_i} \to \mathrm{M}$ is one to one. It is classical that a submonoid $\mathbb{M}$ of $\mathbb{M}(A, \theta)$ has a unique minimal generating set. Hence, a factorization

can be characterized by the family of the generating sets of its components and will be denoted by $\mathbb{F} = (Y_i)_{i\in I}$ instead of $\mathbb{F} = (\mathbb{M}_i)_{i\in I}$ if $Y_i = \mathbb{M}_i - \mathbb{M}_i^2$.

The existence of the root of conjugacy classes allows us to extend Schützenberger's factorization theorem [16] to trace monoids.

Theorem 2.3 *Let $\mathbb{F} = (Y_i)_{i\in J}$ be an ordered family of non-commutative subsets (i.e. for each i and each pair $(x, y) \in Y_i^2$, $x \neq y$ implies $xy \neq yx$) of $\mathbb{M}(A, \theta)$ and $\langle Y_i\rangle$ the submonoid generated by Y_i.*
We consider the following assertions :

1. *The mapping* prod *is into.*
2. *The mapping* prod *is onto.*
3. *Each monoid $\langle Y_i\rangle$ is free. For each conjugacy class C in $\mathbb{M}(A, \theta)$, if C is connected (i.e. if the restriction of the non-commutation graph to the alphabet* $\mathrm{Alph}(t) = \{a \in A | t = uav\}$ *is a connected graph) then it exists an unique $i \in J$ such that $C \cap \langle Y_i\rangle \neq \emptyset$ and in this case $C \cap \langle Y_i\rangle$ is a conjugacy class of $\langle Y_i\rangle$. If C is not connected, for each $i \in J$, $C \cap \langle Y_i\rangle = \emptyset$.*

Two of the previous assertions imply the third.

Proof We prove 1) and 2) imply 3) by means of the examination of the series

$$\log \underline{\mathbb{M}(A, \theta)} - \sum \log \underline{\langle Y_i\rangle}. \tag{2}$$

Remarking that 1) and 2) force the set $\mathbb{F}$ to be a factorization, it is easy to show that (2) is a Lie series whose valuation is strictly greater than 1. Hence, if C is a conjugacy class of $\mathbb{M}(A, \theta)$, one has

$$\left(\underline{C}, \sum \log \underline{\langle Y_i\rangle}\right) = \left(\underline{C}, \sum_{l\in Ly(A,\theta)} \log \frac{1}{1-l}\right) \tag{3}$$

where $Ly(A, \theta)$ denotes the set of Lyndon traces (which is a complete factorization of $\mathbb{M}(A, \theta)$ for the standard order [13]) and (,) is the scalar product for which the monomials form an orthonormal family.
If C is not connected (3) implies

$$\left(\underline{C}, \sum_i \sum_m \frac{1}{m}\underline{Y_i^m}\right) = \left(\underline{C}, \sum_l \sum_m \frac{1}{m} l^m\right) = 0 \tag{4}$$

and, the series $\sum_i \sum_m \frac{1}{m} \underline{Y_i^m}$ being positive, one gets $\langle Y_i\rangle \cap C = \emptyset$.
If C is strongly primitive (*i.e.* C is connected and $\sqrt{C} = C$), equality (3) implies

$$\left(\underline{C}, \sum_i \sum_m \frac{1}{m} \underline{Y_i^m}\right) = 1. \tag{5}$$

Let i and m be such that $C \cap Y_i^m \neq \emptyset$. The strong primitivity of $\underline{C}$ implies

$$\left(\underline{C}, \sum_i \sum_m \frac{1}{m} \underline{Y_i^m}\right) \geq 1 \tag{6}$$

and the unicity of Y_i follows. Furthermore, by Card $C \cap Y_i^m = m$, we show that $C \cap Y_i^m$ is a conjugacy class in $\langle Y_i\rangle$.
Now, suppose that C is connected but not primitive and let $p > 1$ such that $C = \{g^p / g \in \sqrt{C}\}$. Let Y_i such that $\sqrt{C} \cap \langle Y_i\rangle \neq \emptyset$ is a conjugacy class in $\langle Y_i\rangle$ (from the previous case, Y_i exists and is unique). One has $C \cap \langle Y_i\rangle = C_j$ where each C_j is a conjugacy class in $\langle Y_i\rangle$. But for each j,

$$\left(\underline{C_j}, \sum_m \frac{1}{m} \underline{Y_i^m}\right) = \frac{1}{p} \tag{7}$$

and from (3) one has

$$\left(\underline{C}, \sum_m \frac{1}{m} \underline{Y_i^m}\right) = \frac{1}{p}. \tag{8}$$

The unicity of Y_i and C_j follows.

The proofs of the two other implications are slight adaptations of the case where $\theta = \emptyset$. The reader can refer to [17] for more details. ■

Denote by $\mathrm{Cont}(\mathbb{F}) = \bigcup_{i\in J} Y_i$ the contents of the factorization $\mathbb{F} = (Y_i)_{i\in I}$. The following result is an extension of a classical result due to Schützenberger [16].

Corollary 2.4 *Let $\mathbb{F}$ be a complete factorization of $\mathbb{M}(A, \theta)$ (i.e. each Y_i is a singleton). Then for each conjugacy class C we have*

$$\mathrm{Card}(C \cap \mathrm{Cont}(\mathbb{F})) = \begin{cases} 1 & \textit{if } C \textit{ is strongly connected} \\ & (\textit{i.e. } C \textit{ is connected and } \sqrt{C} = C) \\ 0 & \textit{otherwise} \end{cases}$$

Hence, complete factorizations of $\mathbb{M}(A,\theta)$ receive the same combinatorics than in the free case. In particular, one recovers that the generating series of a complete factorization is equal to the Hilbert series of the free partially commutative Lie algebra.

3 Transitive Lazard sets

3.1 Complete elimination strings and transitive Lazard sets

Let $\mathcal{A}(2,A)$ be the free (non associative) algebra on A whose product will be denoted by $(.,.)$ (*i. e.* the algebra of the binary trees with leaves in A) .
The canonical morphism $\mathcal{A}(2,A) \to \mathbb{M}(A,\theta)$ which is the identity on A will be called the foliage morphism and denoted by f as in [15]. Remark that $\theta_{\mathbb{M}} = \{(w,w')|ww' = w'w \text{ and } \mathrm{Alph}(w) \cap \mathrm{Alph}(w') = \emptyset\}$ is a commutation relation on $\mathbb{M}(A,\theta)$ ([9]).
In this section, we consider a commutation alphabet (A,θ) and some graphs whose vertices belong in $\mathcal{A}(2,A)$ and such that the edges verify the property that (t_1,t_2) is an edge if and only if $(f(t_1),f(t_2)) \in \theta_{\mathbb{M}}$.
Let $G = (V,E)$ be such a graph, we call a **Elimination String** (**ES**) in G a n-uplet of vertices $(a_1,\cdots,a_n)$ such that for each $i \in [1,n]$ and $v_1,v_2 \in V - \{a_1,\cdots,a_i\}$

$$(v_1,v_2) \in E \Rightarrow (v_1,a_i) \in E \text{ or } (v_2,a_i) \in E \tag{9}$$

The vertex a_1 will be called the **starting point** of the **ES**.
An **ES** $(a_1,\cdots,a_n)$ will be called **complete** (**CES** in the following) if $E = \{a_1,\cdots,a_n\}$. A graph admitting a **CES** will be called **type-H graph**.
Let $n \geq 1$. In the sequel we denote $\mathcal{A}(2,A)^{\leq n}$ the set of the trees with less than n leaves.

Example 3.1 *1. Let us consider the following graph*

$$(A,\theta) = \begin{array}{ccccc} a & - & d & - & e \\ | & & | & & \\ b & - & c & & \end{array}$$

the family (a,d,b,c,e) is a **CES** *of (A,θ) and then it is a type H graph.*

2. The graph

$$(A,\theta) = \begin{array}{ccc} a & - & b \\ c & - & d \end{array}$$

is not a type-H graph.

3.2 Transitive Lazard sets

Let v be a vertex of G, the **H-star** of G for v to the rank $n > 0$ is the graph $G_n^{*v} = (V_n^{*v}, E_n^{*v})$ defined by

$$\left\{ \begin{array}{l} V_n^{*v} = (V \cap \mathcal{A}(2,A)^{\leq n} - v) \cup \{(v'v^m) \in \mathcal{A}(2,A)^{\leq n} | m > 0, (v',v) \notin E\} \\ E_n^{*v} = \{(v_1,v_2) \in (V_n^*)^2 | (f(v_1), f(v_2)) \in \theta_{\mathbb{M}}\} \end{array} \right. . \tag{10}$$

Example 3.2 *For the following graph*

$$(A,\theta) = a - b - c - d$$

then

$$(A_4^{*c}, \theta_4^{*c}) = \begin{array}{ccccc} & & ac & & \\ & & | & & \\ a & - & b & - & ((a,c),c) \quad d \\ & & | & & \\ & & (((a,c),c),c) & & \end{array}$$

Proposition 3.3 *Let G be a type-H graph and a_1 be the starting point of a CES then $G_n^{*a_1}$ is a type-H graph.*

Proof Let $\lambda = (a_1, \cdots, a_m)$ be a **CES** of G. We construct a list Λ of vertices of $G_n^{*a_1} = (V^*, E^*)$ removing a_1 of λ and substituting each a_i such that $(a_i, a_1) \notin E$ for the sequence

$$(a_i, a_1^{m-1}), \ldots, (a_i, a_1), a_i.$$

Let us denote $\Lambda = (a'_1, \cdots, a'_M)$ and suppose that Λ is not a **CES** ; then it exists $\alpha < \beta < \gamma$ such that $(a'_\beta, a'_\gamma) \in E^*$, $(a'_\alpha, a'_\beta) \notin E^*$ and $(a'_\alpha, a'_\gamma) \notin E^*$. Set $a'_\alpha = (a_i, a_1^p)$, $a'_\beta = (a_j, a_1^q)$ and $a'_\gamma = (a_k, a_1^r)$. The construction implies that $i \leq j < k$ and $p = 0$ or $q = 0$. If $i = j$ then $p \neq 0$ which implies $(a_i, a_1) \notin E$ and $r = 0$. Hence, $(a_1, a_k) \notin E$ and $q = 0$. This implies that

$(a_i, a_k) \in E$, $(a_1, a_i) \notin E$ and $(a_1, a_k) \notin E$ and contradicts the fact that λ is a **CES**. If $i < j$, suppose that $r = 0$ (the case $q = 0$ is symmetric), we need to examine several cases

1. If $p = 0$, then $q \neq 0$ (otherwise $(a_i, a_j), (a_i, a_k) \notin E$ and $(a_j, a_k) \in E$ and this contradicts the fact that λ is a **CES**). But, as $(a_k, a_i) \notin E$, $(a_j, a_k) \in E$ and as λ is a **CES**, one has $(a_i, a_j) \in E$ and hence $(a_1, a_i) \notin E$. Finally $(a_1, a_j) \notin E$ contradicts the fact that λ is a **CES**.

2. If $p \neq 0$ and $q = 0$, then, as λ is a **CES**, either $(a_i, a_j) \in E$ or $(a_i, a_k) \in E$. Suppose that $(a_i, a_j) \in E$ (the other case is symmetric), one has $(a_1, a_j) \in E$ (otherwise $(a'_\alpha, a'_\beta) \in E^*$). But, λ being a **CES**, one gets $(a_1, a_k) \in E$ and by $(a'_\alpha, a'_\gamma) \notin E^*$, one obtains $(a_i, a_k) \notin E$. Finally $(a_1, a_j), (a_1, a_i) \notin E$ and $(a_i, a_j) \in E$ contradicts the fact that λ is a **CES**.

3. If $p, q \neq 0$ then, as $(a'_\gamma, a'_\beta) \in E^*$, one as $(a_1, a_k) \in E$ and by $(a'_\alpha, a'_\gamma) \notin E^*$, one obtains $(a_i, a_k) \notin E$. Hence, λ being a **CES**, $(a_i, a_j) \in E$. Hence $(a_1, a_j), (a_1, a_i) \notin E$ and $(a_i, a_j) \in E$ contradicts the fact that λ is a **CES**.

This prove that Λ is a **CES**. ■

Let $G = (A, \theta)$ be a commutation alphabet considered as a graph and $L \subset \mathcal{A}(2, A)$, we will say that L is a **transitive Lazard set** if and only if for each $n > 0$, $L \cap \mathcal{A}(2, A)^{\leq n} = \{s_1, \cdots, s_k\}$ such that it exists $k + 1$ graphs $G_1 = (A_1, \theta_1), \ldots, G_{k+1} = (A_{k+1}, \theta_{k+1})$ satisfying the following conditions:

1. The first graph G_1 is equal to G.
2. The last graph G_{k+1} is empty (i.e. $G_{k+1} = (\emptyset, \emptyset)$)
3. For each $i < k + 1$, $s_i \in A_i$ and s_i is the starting point of a **CES** of G_i.
4. We have $G_{i+1} = (G_i)^{*s_i}_n$.

3.3 Type-H graphs and the transitive factorization theorem

The classical properties of the (non commutative) Lazard sets hold true and can be seen as consequences of the Transitive Factorization theorem [9]. We

recall them here.
Let (A, θ) be a partially commutative alphabet and $B \subset A$. Then $\mathbb{M}(B, \theta_B)$ is the left (resp. right) factor of a bisection of $\mathbb{M}(A, \theta)$. Explicitly,

$$\mathbb{M}(A, \theta) = \mathbb{M}(B, \theta_B).\langle \beta_Z(B) \rangle$$

where $\langle \beta_Z(B) \rangle$ denotes the submonoid generated by the set

$$\beta_Z(B) = \{zw/z \in Z, w \in \mathbb{M}(B, \theta_B), IA(zw) = \{z\}\}$$

and $IA(t) = \{z \in A | t = zw\}$ is the initial alphabet of the trace t.
Let $B \subset A$, we say that B is a **transitively factorizing subalphabet** (TFSA) if and only $\beta_Z(B)$ is a partially commutative code [7]. We proved the following theorem.

Theorem 3.4 *Duchamp-Luque[9]*

1. *Let $B \subset A$. The following assertions are equivalent.*

   (i) The subalphabet B is a TFSA.

   (ii) The subalphabet B satisfies the following condition.
   For each $z_1 \neq z_2 \in Z$ and $w_1, w_2, w_1', w_2' \in \mathbb{M}(A, \theta)$ such that $IA(z_1 w_1) = IA(z_1 w_1') = \{z_1\}$ and $IA(z_1 w_2) = IA(z_2 w_2') = \{z_2\}$ we have

   $$z_1 w_1 z_2 w_2 = z_2 w_2' z_1 w_1' \Rightarrow w_1 = w_1', w_2 = w_2'.$$

   (iii) For each $(z, z') \in Z^2 \cap \theta$, the dependence graph (ie non-commutation) has no partial graph like

   $$z - b_1 - \ldots - b_n - z'.$$

   with $b_1, \ldots, b_n \in B$.

2. *Let (B, Z) be a partition of A*

   (i) We have the decomposition

   $$L(A, \theta) = L(B, \theta_B) \oplus J$$

   where J is the Lie ideal generated (as a Lie algebra) by

   $$\tau_Z(B) = \{[\ldots[z, b_1], \ldots b_n] \quad | \quad z b_1 \ldots b_n \in \beta_Z(B)\}.$$

(ii) The subalgebra J is a free partially commutative Lie algebra if B is a TFSA of A.
(iii) Conversely if J is a free partially commutative Lie algebra with code $\tau_Z(B)$ then B is a TFSA.

Applying theorem 3.4 and taking the inductive limit of the process one obtains.

Proposition 3.5 *Let L be a transitive Lazard set.*

1. *The foliage $f(L)$ is a complete factorization of $\mathbb{M}(A, \theta)$.*

2. *Let Π be the unique morphism $\mathcal{A}(2, A) \to L(A, \theta)$ such that $\Pi(a) = a$ for each letter $a \in A$. Then $\Pi(L)$ is a basis of the Free Lie algebra $L(A, \theta)$.*

Such a factorization will be called Transitive Lazard Factorization (TLF). Not all the trace monoids possess a TLF. For example, in the graph $\begin{smallmatrix} a-b \\ c-d \end{smallmatrix}$ we can not find a **CES**. Nevertheless, the property "having a TLF" is decidable as shown by the following result.

Theorem 3.6 *A trace monoid admits a TLF if and only if its commutation alphabet is a type-H graph.*

Proof It suffices to remark that a trace monoid has a TLF if and only if on can construct a transitive Lazard set from its commutation graph which is a consequence of proposition 3.3. ■

4 Transitive Hall sets

Let us define a **Transitive Hall Set** (**THS**) H as a family of trees $(h)_{h \in H}$ endowed with a total order $<$ such that

1. The family $(f(h))_{h \in H}$ is a complete factorization of $\mathbb{M}(A, \theta)$ (for the reverse order).

2. The set H contains the alphabet A.

3. If $h = (h', h'') \in H - A$ then $h'' \in H$ and $h < h''$.

4. If $h = (h', h'') \in \mathcal{A}(2, A) - A$ then $h \in H$ if and only if the four following assertions are true :
   (a) The two sub-trees h' and h'' belong to H.
   (b) We have the inequality $h' < h''$.
   (c) The foliage of the two sub-trees are not related by the "disjoint commutation" relation (i.e. $(f(h'), f(h'')) \notin \theta_{\mathbb{M}}$).
   (d) Either $h' \in A$ or $h' = (x, y)$ with $y \geq h''$.

Note that in the classic non-commutative theory of Hall set, one gets (1) from the axioms (2), (3) and (4). Here, it is not the case. For example, if we consider the commutation graph

$$a \qquad b - c$$

and a set H such that

$$\begin{array}{rcl} H \cap \mathcal{A}(2, A)^{\leq 3} & = & \{a, (b, a), ((c, a), a), c, ((c, a), c), (c, a), (b, (c, a)), \\ & & ((b, a), c), b, ((b, a), b), (b, a)\}. \end{array}$$

We suppose that trees above are ordered from the right to the left. Clearly, trees listed here verify axioms (2), (3) and (4) of the definition of transitive Hall sets, but we can not complete $H \cap \mathcal{A}(2, A)^{\leq 3}$ to construct a complete factorization as $c.ba$ is a decreasing decomposition of the trace bca which belongs to $f(H)$.

As in the free case we have a perfect correspondence between the notions of transitive Hall and Lazard sets.

Theorem 4.1 *A set L is a transitive Lazard set if and only if it is a transitive Hall set.*

Sketch of the proof A slight adaptation of the non commutative case (see [15]) shows that each transitive Lazard set is a Transitive Hall set.
Let us sketch a proof of the converse. The notion of Transitive Lazard sets is independent of the filtration in the following sense: L is a Lazard set if and only if for each finite closed set E (*ie* $(t_1, t_2) \in E \Rightarrow t_1, t_2 \in E$) denoting $L \cap E = \{s_1, \ldots, s_k\}$ there exist $k + 1$ graphs $G_i = (A_i, \theta_i)$ verifying

1. $G_1 = (A, \theta)$,

2. $G_{k+1} = (\emptyset, \emptyset)$

3. For each $i < k+1$, $s_i \in A_i$ and s_i is the starting point of a CES over G_i.

4. $A_{i+1} = (A_i - s_i) \cup \{(bc_i^n) \in E | n > 0, b \in A_i \text{ and } 0(b, c_i) \notin \theta_i\}$ and $\theta_{i+1} = \theta_{A_{i+1}}$.

The fact that, for each closed set E, any transitive Hall set satisfies the four previous properties will be proved by induction on the cardinal of E. Let H be a THS. If Card $E = 1$ the result is obvious. We suppose now that Card $E > 1$ and we set

$$c = max\{h \in H \cap E\}$$

$$X = \{(ac^n) | a \in A - c, n \geq 0, (a, c) \in \theta\} \cap \{b | (b, c) \in \theta\}$$

$$\theta_X = \theta_{\mathbb{M}} \cap X \times X$$

$$H' = H \cap H \cap \mathcal{A}(2, X) \text{ and } E' = E \cap \mathcal{A}(2, X)$$

As in [15], H' is a transitive Hall set for the alphabet X and $H' \cap E' = H \cap E'$. By induction H' is a transitive Lazard set and one constructs $k+1$ graphs $G_1 = (A_1, \theta_1), \ldots, G_{k+1} = (A_{k+1}, \theta_{k+1})$ verifying (1), (2), (3), and (4). Remark that if $H \cap E' = \{s_1, \ldots, s_n\}$, one has $H \cap E = \{s_0 = c, s_1, \ldots, s_n\}$. Suppose that it exists $(a, b) \in \theta$ such that $(a, c) \notin \theta$ and $(b, c) \notin \theta$ then a quick examination shows that $(f(h))_{\mathsf{H}}$ is not a factorization. Hence, c is the starting point of a CES in $G_0 = (A, \theta)$ and we have $A_1 = (A - c) \cup \{(ac^n) \in E | n > 0, a \in A \text{ and } 0(a, c) \notin \theta\}$. It follows that H and E verify the assertions (1), (2), (3) and (4). This proves that H is a transitive Lazard set. ■

This correspondence is very useful to construct decomposition algorithms. We can construct **standard sequences** of Hall trees $(h_1, \cdots, h_n)$ such that $L(A, \theta)$ for each $i \in [1, n]$ either $h_i \in A$, or $h_i = (h_i', h_i'')$ with $h_i'' \geq h_{i+1}, \ldots, h_n$. In a standard sequence an **ascent** is an index i such that $h_i < h_{i+1}$ and a **legal ascent** is an ascent i such that $h_{i+1} \geq h_{i+2}, \cdots, h_n$ (these definitions are due to Schützenberger [17]). Let s be a standard sequence and i a legal ascent. We write $s \rightarrow s'$ if $s' = (h_1, \cdots, h_{i-1}, (h_i, h_{i+1}), h_{i+2}, \cdots, h_n)$ when $(f(h_i), f(h_{i+1})) \notin \theta_{\mathbb{M}}$ and $s' = (h_1, \cdots, h_{i-1}, h_{i+1}, h_i, h_{i+2}, \cdots, h_n)$ otherwise. The transitive closure $\xrightarrow{*}$ of $\rightarrow$ is such that, for each standard sequence it exists a unique decreasing standard sequence s' such that $s \xrightarrow{*} s'$.

Using this property on a sequence of letters, we obtain an algorithm which allows to find the factorization of a trace in a decreasing concatenation of Hall traces.

Example 4.2 *Let us consider the following commutation alphabet:*

$$(A, \theta) = a - b - c - d.$$

and let H *be a transitive hall set such that*

$H \cap \mathcal{A}(2, A)^{\leq 3} = \{c, b, a, (a, c), ((a, c), c), (a, (a, c)), (d, b), ((d, b), b), (d, (b, a)), (d, (a, c)), (d, a), ((d, a), a), (d, (d, b)), (d, (d, a)), d\}.$

We can compute the factorization of the word bcaccbdbddad in the following way

$$\begin{array}{c} (b, c, a, c, c, b, d, b, d, d, a, d) \\ \downarrow \\ (b, c, a, c, c, b, d, b, d, (d, a), d) \\ \downarrow \\ (b, c, a, c, c, b, d, b, (d, (d, a)), d) \\ \downarrow \\ (b, c, (a, c), c, b, (d, b), (d, (d, a)), d) \\ \downarrow \\ (b, c, ((a, c), c), b, (d, b), (d, (d, a)), d) \\ \downarrow \\ (b, c, b, ((a, c), c), (d, b), (d, (d, a)), d) \\ \downarrow \\ (c, b, b, ((a, c), c), (d, b), (d, (d, a)), d) \end{array}$$

which gives $bcaccbdbddad = c.b.b.acc.db.dda.d.$

Let $s = (h_1, \cdots, h_n)$ be a standard sequence and i a legal ascent, we define

$$\lambda_i(s) = (h_1, \ldots, h_{i-1}, (h_i, h_{i+1}), h_{i+2}, \ldots, h_n) \tag{11}$$

and

$$\rho_i(s) = (h_1, \cdots, h_{i-1}, h_{i+1}, h_i, h_{i+2}, \cdots, h_n). \tag{12}$$

The **derivation tree** of s is the tree $T(s)$ satisfying the following

1. if s is a decreasing sequence then $T(s)$ is only the root labeled s

2. otherwise, we consider the greatest legal ascent i of s. Then

   (a) if $(h_i, h_{i+1}) \in \theta_{\mathbb{M}}$, the root of the tree $T(s)$ is s and $T(s)$ has only one sub-tree $T(\rho_i(s))$.

   (b) otherwise, the root of $T(s)$ is s, the left sub-tree of $T(s)$ is $T(\lambda_i(s))$ and the right sub-tree of $T(s)$ is $T(\rho_i(s))$

If we denote $[s] = [h_1] \cdots [h_n]$, one obtains

$$[s] = \sum_{s' \in \mathsf{F}(T(s))} [s'] \tag{13}$$

where $\mathsf{F}(T(s))$ denotes the set of the leaves of $T(s)$. Applying this equality to sequences of words, one gets an algorithm allowing to decompose a polynomial in the PBW basis associated to a transitive Hall set.

Example 4.3 *We use the transitive Hall set defined in the example 4.2. One has for example:*

$$\begin{tikzcd}
 & (b,c,a,c,c,b,d) \arrow[dl] \arrow[dr] & & \\
(b,c,c,a,c,b,d) \arrow[d] \arrow[dr] & & (b,c,(a,c),c,b,d) \arrow[d] \arrow[dr] & \\
(b,c,c,c,a,b,d) \arrow[d] & (b,c,c,(a,c),b,d) \arrow[d] & (b,c,c,(a,c),b,d) \arrow[d] & (b,c,((a,c),c),b,d) \arrow[d] \\
(b,c,c,c,b,a,d) \arrow[d] & (b,c,c,b,(a,c),d) \arrow[d] & (b,c,c,b,(a,c),d) \arrow[d] & (b,c,b,((a,c),c),d) \arrow[d] \\
(c,b,c,c,b,a,d) \arrow[d] & (c,b,c,b,(a,c),d) \arrow[d] & (c,b,c,b,(a,c),d) \arrow[d] & (c,c,b,((a,c),c),d) \\
(c,c,b,c,b,a,d) \arrow[d] & (c,c,b,b,(a,c),d) & (c,c,b,b,(a,c),d) & \\
(c,c,c,b,b,a,d) & & &
\end{tikzcd}$$

Which allows to write

$$bcaccbd = c.c.c.b.b.a.d + 2c.c.b.b.[a,c].d + c.b.b.[[a,c],c].d.$$

5 Conclusion

The stable concept of transitive factorization allows the adaptation of the Hall machinery as it has been explicited here. The construction is characteristic free. It would be interesting to investigate such constructions in the case of p-commutations [4, 10].

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